RISK MANAGEMENT AND DYNAMIC PORTFOLIO SELECTION WITH STABLE PARETIAN DISTRIBUTIONS

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ABSTRACT: This paper assesses stable Paretian models in portfolio theory and risk management. We describe an investor's optimal choices under the assumption of non-Gaussian distributed equity returns in the domain of attraction of a stable law. In particular, we examine dynamic portfolio strategies with and without transaction costs in order to compare the forecasting power of discrete-time optimal allocations obtained under different stable Paretian distributional assumptions. We also consider a conditional extension of the stable Paretian approach and compare the model with others that consider different distributional assumptions. Finally, we empirically evaluate the forecasting power of the model for predicting the value at risk of a heavy-tailed return series.
KEY WORDS: stable Paretian distributions, multi-period portfolio choice, value at risk, dynamic portfolio strategies.
JEL CLASSIFICATION: G11, G14, C61

1. Introduction

In this paper, we propose some stable Paretian models for optimal portfolio selection and for quantifying the risk of a given portfolio. After examining the multi-period optimal portfolio problems under different distributional assumptions, we propose an *ex-ante* and an *ex-post* empirical comparison between the stable Paretian approach and a moment-based one. We then discuss how to use the stable Paretian model to compute the value at risk (VaR) of a given portfolio.

It is well known that asset returns are not uniquely determined by their mean and variance. Numerous empirical studies, beginning with the works of Mandelbrot (1963a, 1963b, 1967) and Fama (1963, 1965a, 1965b), have refuted the commonly accepted view that financial returns are normally distributed.[1] In this paper, we examine the implications of different distributional hypotheses for dynamic portfolio strategies of investors. In particular, we compare the performance of dynamic strategies based on a stable Paretian model and on a moment-based model.

The literature on multi-period portfolio selection has focused on maximizing expected utility functions of terminal wealth and/or multi-period consumption. In contrast to the focus of classical multi-period approaches, we generalize the mean-variance analysis suggested by Li and Ng (2000), providing a three-parameter formulation of optimal dynamic portfolio selection. These alternative multi-period approaches are consistent with the admissible optimal portfolio choices of risk-averse investors. In particular, we develop analytical optimal portfolio policies for the multi-period mean-dispersion-skewness formulation. In order to compare a moment-based three-parameter portfolio model and the stable Paretian dynamic model, we analyze several investment allocation problems.

The primary contribution of the empirical comparison presented in this paper is the analysis of the impact of the distributional assumptions on multi-period asset allocation decisions.

[1] See Rachev and Mittnik (2000) and the reference therein.

Thus, we propose comparing the different distributional assumptions considering either that (1) the vectors of returns are independent and unlimited short sales and no transaction costs are allowed, or (2) the vectors of returns are time dependent and there exist both portfolio constraints and transaction costs. For these two comparisons we use both historical data and simulated data. For this purpose, we analyze some allocation problems for non-satiable risk-averse investors with different risk-aversion coefficients. We determine the *ex-ante* and *ex-post* multi-period efficient frontiers given by the minimization of the conditional dispersion measures. Each investor, characterized by his utility function, will prefer the model which maximizes his expected utility on the efficient frontier. The portfolio policies obtained with this methodology represent the optimal choices for the different approaches for an investor. Therefore, we examine the differences in optimal strategies for an investor under the stable and the moment-based distributional hypothesis.

In addition, we propose an *ex-ante* and an *ex-post* comparison between the parametric-portfolio selection models proposed assuming that no short sales and transaction costs are allowed. Thus we assess these models considering that every day each investor recalibrates his portfolio in order to maximize his expected utility on a three-parametric efficient frontier. Finally, we assess the power of these models for forecasting VaR of a given portfolio.

2. Three-parameter portfolio selection models with and without short-sale constraints

In this section, we analyze a discrete-time extension of the Li and Ng (2000) problem, assuming the vectors of returns are either independent or dependent. As for the Li-Ng model, we get a closed-form solution of optimal choices if vectors of returns are independent. In particular, we consider the optimal allocation among $n+1$ assets: n of those assets are risky assets with stable distributed risky returns $z_{t_j} = [z_{1,t_j}, ..., z_{n,t_j}]'$ over the time period $[t_j, t_{j+1})$ and the $(n+1)th$ asset is risk-free with returns $z_{0,t}$ for $t = t_0, t_1, ..., t_{T-1}$.

Let W_{t_j} be the wealth of the investor at the beginning of the period $[t_j, t_{j+1})$, and let x_{i,t_j}

$i = 1, ..., n$; $t_j = t_0, t_1, ..., t_{T-1}$ (with $t_0 = 0$ and $t_i < t_{i+1}$) be the amount invested in the *i-th* risky asset at the beginning of the period $[t_j, t_{j+1})$. x_{0,t_j}; $t_j = 0, t_1, ..., t_{T-1}$ is the amount invested in the risk-free asset at the beginning of the period $[t_j, t_{j+1})$.

2.1 *A three-parameter extension of the Li-Ng model*

Li and Ng (2000) have proposed an analytical solution to the dynamic mean-variance portfolio selection problem when the vectors of risky returns z_t are statistically independent and the first two moments are finite. In their analysis, they assume that the amounts invested in the assets at the beginning of each period $[t_j, t_{j+1})$ could be random variables. In contrast to the Li-Ng model, we assume that the multi-period portfolio policies for the risky assets $x_{t_j} = [x_{1,t_j}, ..., x_{n,t_j}]'$ for any j are deterministic variables of the problem and the wealth invested in the risk-free asset at time t_j is given by $x_{0,t_j} = W_{t_j} - x'_{t_j} e$ where $e = [1, ..., 1]'$ (and, clearly, it is a random variable). In the following analysis, we assume that the wealth process is uniquely determined by three parameters as in the model proposed by Ortobelli et al. (2004): mean, dispersion, and skewness. In particular, we assume:

a) the initial wealth $W_0 = \sum_{i=0}^{n} x_{i,0}$ is known and the vectors of returns $z_t = [z_{1,t}, ..., z_{n,t}]'$ are time independent [2] of any time $t = t_0, t_1, ..., t_{T-1}$;

b) the returns $z_{i,t}$ follow the model

$$z_{i,t} = \mu_{i,t} + b_{i,t} Y_t + \varepsilon_{i,t} \tag{1}$$

where $\mu_{i,t}$ is the mean of $z_{i,t}$, $Y_t \sim S_{\alpha_2}(\sigma_Y, \beta_Y, 0)$ for $t = t_0, t_1, ..., t_{T-1}$ are independent iden-

[2] The vectors of returns are not necessarily identically distributed since the vector of expected returns and the vectors b_t are not necessarily constant. This could be the case when we assume (only for brief periods) that the length of the periods of analysis, $t_{j+1} - t_j$, is constant varying j. When we consider daily or weekly returns, we can adopt either continuously compounded returns $z_{i,t} = \ln\left(\frac{S_{i,t+1}}{S_{i,t}}\right)$ or the returns $z_{i,t} = \frac{S_{it+1} - S_{it}}{S_{it}}$ (where $S_{i,t}$ is the price of the *i-th* asset at time t). As a matter of fact, daily or weekly continuously compounded returns approximate well enough the returns $\frac{S_{it+1} - S_{it}}{S_{it}}$ and we generally do not observe material differences in the portfolio strategies obtained with the two alternative definitions (see, among others, Biglova et al., 2004). In addition, the empirical evidence shows that daily or weekly returns are very often in the domain of attraction of stable laws (see Rachev and Mittnik, 2000) and the reference therein).

tically distributed (i.i.d.) α_2-stable Paretian asymmetric distributed ($\beta_Y \neq 0$, $\alpha_2 \in (1,2)$). Moreover, Y_t is independent of α_1-stable sub-Gaussian distributed vectors of disturbances $\varepsilon_t = [\varepsilon_{1,t}, ..., \varepsilon_{n,t}]'$ and $\alpha_1 \in (1,2]$. Vectors of disturbances ε_t are statistically independent for any $t = t_0, t_1, ..., t_{T-1}$. Observe that the assumption that the vector of disturbances ε_t is elliptical distributed as an α_1-stable sub-Gaussian implies that the vector of returns $z_t = \mu_t + b_t Y_t + \varepsilon_t$ describes a three-fund separation model (see Ross, 1978; and Simaan, 1993). Under these assumptions, the dependence structure of returns is essentially described by the dispersion matrix Q_t of the elliptical vector of disturbances ε_t.

Under these assumptions, the vectors of returns admit finite all the positive fractional moments with exponent below the $\min(\alpha_1, \alpha_2)$. Moreover, z_t admits the following characteristic function $\Phi_{z_t}(u) = E(\exp(iu'z_t))$:

$$\Phi_{z_t}(u) = \exp\left(-\left(u'Q_t u\right)^{\alpha_1/2} - \left|u'b_t\sigma_Y\right|^{\alpha_2}\left(1 - i\left(u'b_t\sigma_Y\right)^{\langle\alpha_2\rangle}\beta_Y \tan\frac{\pi\alpha_2}{2}\right) + iu'\mu_t\right)$$

where $x^{\langle\alpha\rangle} = sgn(x)\left|x\right|^{\alpha}$, $b_t = [b_{1,t}, ..., b_{n,t}]'$, $\mu_t = E(z_t)$, Q_t is the definite positive dispersion matrix associated with the vector of disturbances $\varepsilon_t = [\varepsilon_{1,t}, ..., \varepsilon_{n,t}]'$ at time t, and σ_Y, β_Y are respectively the scale and the skewness parameter of the centered equity return Y (independent of ε_t). Considering that the wealth at each time is given by

$$\begin{aligned} W_{t_{k+1}} &= \textstyle\sum_{i=0}^{n} x_{i,t_k}(1 + z_{i,t_k}) = \\ &= (1 + z_{0,t_k})W_{t_k} + x'_{t_k}p_{t_k} \quad k = 0, 1, 2, ..., T-1 \end{aligned}$$

where $p_{t_i} = [p_{1,t_i}, ..., p_{n,t_i}]'$ is the vector of excess of returns $p_{k,t_i} = z_{k,t_i} - z_{0,t_i}$, then we can write the final wealth as follows:

$$\begin{aligned} W_{t_T} &= W_0 \textstyle\prod_{k=0}^{T-1}(1 + z_{0,t_k}) + \\ &+ \textstyle\sum_{i=0}^{T-2} x'_{t_i}p_{t_i}\prod_{k=i+1}^{T-1}(1 + z_{0,t_k}) + x'_{t_{T-1}}p_{t_{T-1}} \end{aligned} \tag{2}$$

for any fixed initial wealth W_0. Since the multi-period portfolio policies in the risky assets x_{t_j} are deterministic variables, then the condtional expectation taken at the period t_0 [3] of the final wealth W_{t_T} is given by

$$E_{t_0}(W_{t_T}) = W_0 \prod_{k=0}^{T-1}(1+z_{0,t_k}) + \\ + \sum_{i=0}^{T-2} x'_{t_i} E_{t_0}(p_{t_i}) \prod_{k=i+1}^{T-1}(1+z_{0,t_k}) + x'_{t_{T-1}} E_{t_0}(p_{t_{T-1}}).$$

Moreover, considering that the final wealth is determined by the relationship given by (2) and the vectors of returns follow the stable law given by (1), then the final wealth W_{t_T} maintains the same distributional structure of the returns:

$$W_{t_T} \stackrel{d}{=} W_0 \prod_{k=0}^{T-1}(1+z_{0,t_k}) + \sum_{i=0}^{T-2} x'_{t_i} E_{t_0}(p_{t_i}) \prod_{k=i+1}^{T-1}(1+z_{0,t_k}) + \\ + x'_{t_{T-1}} E_{t_0}(p_{t_{T-1}}) + Y\left(\sum_{i=0}^{T-2} x'_{t_i} b_{t_i} \prod_{k=i+1}^{T-1}(1+z_{0,t_k}) + x'_{t_{T-1}} b_{t_{T-1}}\right) + \\ + \sum_{i=0}^{T-2} x'_{t_i}\varepsilon_{t_i} \prod_{k=i+1}^{T-1}(1+z_{0,t_k}) + x'_{t_{T-1}}\varepsilon_{t_{T-1}} = E_{t_0}(W_{t_T}) + A_x Y + \Psi_x$$

where $A_x = \sum_{i=0}^{T-2} x'_{t_i} b_{t_i} \prod_{k=i+1}^{T-1}(1+z_{0,t_k}) + x'_{t_{T-1}} b_{t_{T-1}}$ is a deterministic variable, $Y \sim S_{\alpha_2}(\sigma_Y, \beta_Y, 0)$ is α_2-stable Paretian distributed independent of the random variable $\Psi_x = \sum_{i=0}^{T-2} x'_{t_i}\varepsilon_{t_i} \prod_{k=i+1}^{T-1}(1+z_{0,t_k}) + x'_{t_{T-1}}\varepsilon_{t_{T-1}}$ (i.e., the sum of α_1-stable independent random variables). The above expression is a consequence of our assumption of independence among the vectors of returns. As a matter of fact, the above equality in distribution is easily obtained using the characteristic functions of random variables $x'_{t_i} p_{t_i}$. Therefore, the final wealth W_{t_T} is a linear combination of two independent stable laws Y (α_2-stable distributed) and Ψ_x is α_1-stable symmetric distributed with null mean and the conditional dispersion $\sigma_{(x'_{t_i}\varepsilon_{t_i})}$ taken at period t_0 defined

[3] Observe that when the vectors of returns are statistically independent (i.e., p_{t_i} is independent of sigma algebra of events at time t_0 generated by log prices in t_0), the condtional expectation taken at the period t_0 is equal to the mean, i.e., $E_{t_0}(p_{t_i}) = E(p_{t_i})$. In particular, under our assumptions we have $E_{t_0}(f(p_{t_i})) = E(f(p_{t_i}))$ for any continuous function f.

by

$$\sigma^{\alpha_1}_{(x'_{t_i}\varepsilon_{t_i})} = \sum_{i=0}^{T-2} \left(x'_{t_i} Q_{t_i} x_{t_i}\right)^{\alpha_1/2} \left(\prod_{k=i+1}^{T-1} (1+z_{0,t_k})\right)^{\alpha_1} + \left(x'_{t_{T-1}} Q_{t_{T-1}} x_{t_{T-1}}\right)^{\alpha_1/2}.$$

Observe that the assumption of independence among the vectors of disturbances is used to find the above formula of scale parameter $\sigma^{\alpha_1}_{(x'_{t_i}\varepsilon_{t_i})}$ (see Samorodnitsky and Taqqu, 1994). Thus, as shown in the Appendix, when unlimited short sales are allowed, any risk-averse investor will choose one of the multi-portfolio policy solutions of the following optimization problem for some m, v, and W_0:

$$\begin{array}{c} \min\limits_{\{x_{t_j}\}_{j=0,1,\ldots,T-1}} \frac{1}{2}\sigma^{\alpha_1}_{(x'_{t_i}\varepsilon_{t_i})} \\ \text{s. t. } E_{t_0}(W_{t_T}) = m; \\ \sum_{i=0}^{T-2} x'_{t_i} b_{t_i} \prod_{k=i+1}^{T-1}(1+z_{0,t_k}) + x'_{t_{T-1}} b_{t_{T-1}} = v \end{array} . \quad (3)$$

Imposing the first-order conditions on the Lagrangian

$$L(x_{t_j}, \lambda_1, \lambda_2) = \frac{1}{2}\sigma^{\alpha_1}_{(x'_{t_i}\varepsilon_{t_i})} - \lambda_1(E_{t_0}(W_{t_T}) - m) - \lambda_2(A_x - v)$$

all the multi-portfolio policy solutions of problem (3) are given by:

$$\begin{array}{c} x_{t_j} = \left(\frac{2}{\alpha_1}\right)^{\frac{1}{(\alpha_1-1)}} \frac{\left(\left(\lambda_1 E(p_{t_j})+\lambda_2 b_{t_j}\right)' Q^{-1}_{t_j}\left(\lambda_1 E(p_{t_j})+\lambda_2 b_{t_j}\right)\right)^{\frac{2-\alpha_1}{(\alpha_1-1)2}}}{B_{j+1}} \times \\ \times Q^{-1}_{t_j}\left(\lambda_1 E(p_{t_j}) + \lambda_2 b_{t_j}\right) \\ \forall j = 0, 1, \ldots, T-2 \\ x_{t_{T-1}} = \left(\left(\lambda_1 E(p_{t_{T-1}}) + \lambda_2 b_{t_{T-1}}\right)' Q^{-1}_{t_{T-1}}\left(\lambda_1 E(p_{t_{T-1}}) + \lambda_2 b_{t_{T-1}}\right)\right)^{\frac{2-\alpha_1}{(\alpha_1-1)2}} \times \\ \times \left(\frac{2}{\alpha_1}\right)^{\frac{1}{(\alpha_1-1)}} Q^{-1}_{t_{T-1}}\left(\lambda_1 E(p_{t_{T-1}}) + \lambda_2 b_{t_{T-1}}\right), \end{array} \quad (4)$$

where $B_i = \prod_{k=i}^{T-1}(1+z_{0,t_k})$ and λ_1, λ_2 are uniquely determined by the following relations

$$\begin{array}{c} \sum_{i=0}^{T-2} x'_{t_i} b_{t_i} B_{i+1} + x'_{t_{T-1}} b_{t_{T-1}} = v \\ \sum_{i=0}^{T-2} x'_{t_i} E_{t_0}(p_{t_i}) B_{i+1} + x'_{t_{T-1}} E_{t_0}(p_{t_{T-1}}) = m - W_0 B_0 \end{array} .$$

Moreover, we can represent the dispersion of final wealth disturbance Ψ_x as a function of the Lagrangian coefficients λ_1, λ_2, i.e.,

$$\sigma^{\alpha_1}_{(x'_{t_i}\varepsilon_{t_i})} = \sum_{j=0}^{T-1} \left(\left(\frac{2}{\alpha_1}\right)^2 \left(\lambda_1 E_{t_0}(p_{t_j}) + \lambda_2 b_{t_j}\right)' Q_{t_j}^{-1} \left(\lambda_1 E_{t_0}(p_{t_j}) + \lambda_2 b_{t_j}\right) \right)^{\frac{\alpha_1}{2(\alpha_1-1)}} .$$

Besides, the wealth invested in the risk-free asset at the beginning of the period $[t_k, t_{k+1})$ is the deterministic wealth $W_0 - x'_0 e$ in t_0, while, for any $k \geq 1$, it is given by the random variable $W_{t_k} - x'_{t_k} e$, where $W_{t_1} = (1+z_{0,0})W_0 + x'_0 p_0$ and for any $j \geq 2$

$$W_{t_j} = W_0 \prod_{k=0}^{j-1}(1+z_{0,t_k}) + \sum_{i=0}^{j-2} x'_{t_i} p_{t_i} \prod_{k=i+1}^{j-1} (1+z_{0,t_k}) + x'_{t_{j-1}} p_{t_{j-1}} \quad (5)$$

In particular, when the vector $\varepsilon_t = [\varepsilon_{1,t}, ..., \varepsilon_{n,t}]'$ is Gaussian distributed (i.e., $\alpha_1 = 2$), we obtain the following analytical solution to the optimization problem (3)

$$\begin{array}{c} x_{t_j} = \frac{(m-W_0B_0)A - vD}{B_{j+1}(AC-D^2)} Q_{t_j}^{-1} E_{t_0}(p_{t_j}) + \frac{vC - (m-W_0B_0)D}{B_{j+1}(AC-D^2)} Q_{t_j}^{-1} b_{t_j} \\ \forall j = 0, 1, ..., T-2 \\ x_{t_{T-1}} = \frac{(m-W_0B_0)A - vD}{AC-D^2} Q_{t_{T-1}}^{-1} E_{t_0}(p_{t_{T-1}}) + \frac{vC - (m-W_0B_0)D}{AC-D^2} Q_{t_{T-1}}^{-1} b_{t_{T-1}} \end{array} , \quad (6)$$

where

$$A = \sum_{i=0}^{T-1} b_{t_i}' Q_{t_i}^{-1} b_{t_i},$$
$$B_i = \prod_{k=i}^{T-1} (1 + z_{0,t_k}),$$
$$C = \sum_{i=0}^{T-1} E_{t_0}(p_{t_i})' Q_{t_i}^{-1} E_{t_0}(p_{t_i})$$
$$\text{and } D = \sum_{i=0}^{T-1} E_{t_0}(p_{t_i})' Q_{t_i}^{-1} b_{t_i}.$$

We obtain the portfolio policies given by (6) even when the vector of disturbances ε_t is elliptical distributed with finite variance and the index Y is an asymmetric random variable with finite third moment. Under this assumption, the variance of final wealth disturbance Ψ_x is a function of m and v. That is:

$$\sigma^2_{\left(x_{t_i}' \varepsilon_{t_i}\right)} = \frac{A\left(m - W_0 B_0\right)^2 + v^2 C - 2v\left(m - W_0 B_0\right) D}{AC - D^2}.$$

We call this approach that assumes disturbances with finite variance the *moment-based approach* in order to distinguish it from the stable Paretian one with $\alpha_1 < 2$. In both cases (stable non-Gaussian and moment-based approaches), the three-fund separation property holds because the multi-portfolio policies in the risky assets x_{t_j} are spanned by vectors $Q_{t_j}^{-1} E_{t_0}(p_{t_j})$, $Q_{t_j}^{-1} b_{t_j}$ for any time t_j. Moreover, simple empirical applications of these formulas show that the implicit term structure $z_{0,t}$ for $t = t_0, t_1, ..., t_{T-1}$ could determine major differences in the portfolio weights of the same strategy and different periods. As a matter of fact, when the interest rates implicit in the term structure are growing (decreasing), investors are more (less) attracted to invest in the risk-free asset in future periods.

2.2 *A three-parameter model with conditional distributions*

In the above three-parameter models, we have used three strong assumptions: (1) unlimited short sales are allowed; (2) time independence of return vectors; and (3) no transaction costs are considered.

All these assumptions serve to determine the closed-form solutions to dynamic portfolio selection problems of risk-averse investors. Clearly, we can relax the assumptions, but then we would not obtain a closed-form solution for the efficient portfolio policies. In order to overcome the limits of the previous model, we assume:

1. The multi-period portfolio policies $x_{t_j} = [x_{1,t_j}, ..., x_{n,t_j}]'$ are deterministic variables of the problem for any j, and the random wealth $W_{t_j} - x'_{t_j}e$ is the wealth invested in the risk-free asset at time t_j. Moreover the initial wealth $W_0 = \sum_{i=0}^{n} x_{i,0}$ is known.

2. The vectors of returns z_{t+1} follow the following model

$$z_{i,t+1} = \mu_{i,t+1} + b_{i,t+1}Y_{t+1} + \varepsilon_{i,t+1}. \tag{7}$$

where $\mu_{i,t+1}$ is the mean of $z_{i,t+1}$, Y_{t+1} for $t = 0, 1, ..., T$ are independent identically distributed asymmetric centered random variables independent of the vector of disturbances $\varepsilon_{t+1} = [\varepsilon_{1,t+1}, ..., \varepsilon_{n,t+1}]'$. We assume that vector ε_{t+1} is elliptical distributed $Ell(0, Q_{t+1/t})$ conditional on the knowledge of a predictible dispersion matrix $Q_{t+1/t} = [\sigma^2_{ij,t+1/t}]$. Under these assumptions, if the length of the periods of analysis is constant, say $t_{j+1} - t_j = 1$, the wealth W_{t_k} at time $t_k = t + k$ (k positive integer) is still expressed by formula (5), but we generally do not know its conditional distribution at time t_k except for $k = 1$. When k=1, the characteristic function of vector z_{t+1} conditioned at time t is given by:

$$E_t(\exp(iu'z_{t+1}) = \exp(iu'\mu_t)\, E_t(\exp(iu'\varepsilon_{t+1}))E_t(\exp(iu'b_{t+1}Y_{t+1})). \tag{8}$$

Therefore, at any time t, risk-averse investors will minimize the conditional portfolio dispersion for a fixed conditional mean and a fixed deterministic parameter $x'_t b_t$. That is, when no short sales are allowed, risk-averse investors should choose a portfolio solution to the problem:

$$\min_{x_t} x_t' Q_{t/t-1} x_t$$
$$\text{s. t. } x_t'\mu_t + (W_t - x_t'e)\, z_{0,t} = m;$$
$$x_t' b_t = v;\ W_t = \textstyle\sum_{i=0}^n x_{i,t};\ x_{i,t} \geq 0.$$

for a known inital wealth W_t and some fixed parameters m and v. Clearly, in any dynamic portfolio decision we should also consider the transaction costs that depend on the choices done in the previous period. We discuss this problem in Section 3.3.

As suggested by Lamantia, et al. (2006) and Ortobelli et al. (2004), we can assume that the elements of the dispersion matrix follow an exponential-weighted moving average model (EWMA). Therefore, if the vectors of disturbances ε_t admit finite the first two moments, the elements $\sigma^2_{ij,t+1/t}$ follow the rule:

$$\sigma^2_{ij,t+1/t} = (1-\lambda)\left(\widetilde{z}_{j,t} - b_{j,t}Y_t\right)\left(\widetilde{z}_{i,t} - b_{i,t}Y_t\right) + \lambda \sigma^2_{ij,t/t-1}. \tag{9}$$

where $\widetilde{z}_{i,t} = z_{i,t} - \mu_{i,t}$ and λ is a parameter (decay factor) that regulates the weighting on past covariation parameters. If vector ε_t is conditional α_1-stable sub-Gaussian distributed, it admits finite all the positive fractional moments with exponent below α_1 ($\alpha_1 \in (1,2]$). Thus, as a consequence of stable Paretian covariation properties (see Samorodnitsky and Taqqu, 1994), the elements of the dispersion matrix $Q_{t+1/t} = [\sigma^2_{ij,t+1/t}]$ should follow the formulas:

$$\begin{gathered}
\sigma^p_{ii,t+1/t} = E_t(|\varepsilon_{i,t+1}|^p)A(p) = (1-\lambda)\left|\widetilde{z}_{i,t} - b_{i,t}Y_t\right|^p A(p) + \lambda\sigma^p_{ii,t/t-1} \\
B_{ij,t+1/t}(p) = E_t(|\varepsilon_{i,t+1} + \varepsilon_{j,t+1}|^p)A(p) = \\
= (1-\lambda)\left|(\widetilde{z}_{i,t} - b_{i,t}Y_t) + (\widetilde{z}_{j,t} - b_{j,t}Y_t)\right|^p A(p) + \lambda B_{ij,t/t-1}(p) \\
\sigma^2_{ij,t+1/t} = \frac{\left(B_{ij,t+1/t}(p)\right)^{2/p} - \sigma^2_{ii,t+1/t} - \sigma^2_{jj,t+1/t}}{2},
\end{gathered} \tag{10}$$

where $A(p) = \frac{\Gamma\left(1-\frac{p}{2}\right)\sqrt{\pi}}{2^p\Gamma\left(1-\frac{p}{\alpha}\right)\Gamma\left(\frac{p+1}{2}\right)}$, for $p \in (0, \alpha_1)$. In particular, we could assume that Y_t is α_2-

stable Paretian distributed with parameters $\beta_Y \neq 0$, $\alpha_2 \in (1,2)$, i.e. $Y_t \sim S_{\alpha_2}(\sigma_Y, \beta_Y, 0)$, and ε_t is conditional α_1-stable sub-Gaussian distributed.

Moreover, we can use the previous model to compute VaR or any related measure such as the average value at risk (AVaR) of a given portfolio by making specific assumptions about the type of elliptical distribution $Ell(0, Q_{t+1/t})$ (e.g., a multivariate Student t, or a multivariate Gaussian or many others) and of the asymmetric distribution Y_t. For example, when $\alpha := \alpha_1 = \alpha_2$, the centered continuously compounded return vector $\widetilde{z}_{t+1} = [\widetilde{z}_{1,t+1}, ..., \widetilde{z}_{n,t+1}]'$ is conditional α-stable distributed and the forecast $(1-\theta)$ VaR of a portfolio $\widetilde{z}_{p,t} = w'\widetilde{z}_t = \sum_{i=1}^{n} w_i \widetilde{z}_{i,t}$ in the period $[t, t+1]$ is given by the corresponding $(1-\theta)$ percentile of the α-stable distribution $S_\alpha\left(\sigma_{p,t+1/t}, \beta_{p,t+1/t}, 0\right)$, where

$$\sigma_{p,t+1/t} = \left(\left(w'Q_{t+1/t}w\right)^{\alpha/2} + |w'b\sigma_Y|^{\alpha}\right)^{1/\alpha}$$

is the forecasted volatility and

$$\beta_{p,t+1/t} = \frac{|w'b\sigma_Y|^{\alpha}\, sgn(w'b)\beta_Y}{\left(w'Q_{t+1/t}w\right)^{\alpha/2} + |w'b\sigma_Y|^{\alpha}}$$

is the forecasted skewness parameter.

As discussed by Simaan (1993) and Ortobelli et al. (2005), when we consider a three-fund separation model, the solution of any allocation problem depends on the choice of the asymmetric random variable Y. Clearly, one should expect that the optimal allocation will differ when one assumes that asset returns are in the domain of attraction of a stable law or that they depend on another three-parameter model. In order to examine the impact of these different distributional assumptions, in the next section we compare the performance of the two models.

3. A comparison among parametric dynamic strategies

In this section, after a preliminary analysis of our dataset, we evaluate and compare the relative performances of the fund-separation portfolio models previously presented. In particular, we propose an *ex-ante* and an *ex-post* comparison between the stable non-Gaussian, the moment-based approaches, and approaches based on simulating scenarios of disturbances either with a multivariate Student t or with a multivariate stable sub-Gaussian distribution. In our comparisons, we assume dynamic portfolio choice strategies either when short sales are allowed and the vectors of returns are independent or when the returns are time dependent and transaction cost constraints and no short sales are allowed.

For both comparisons, we assume that investors recalibrate their portfolio daily. Thus, we analyze optimal dynamic strategies during a period of about five years (1,250 trading days) among a risk-free asset proxied by the 30-day Eurodollar CD (and offering a rate of one-month Libor), and 24 developed country stock market indices. The countries are shown in the first column of Table 1. The stock indices are those that are or have been part of the MSCI World Index since 1988. The historical returns for all of the stock indices cover the period from 1/5/1988 to 5/26/2009 for a total of 5,579 observations. We split the historical return data series into two parts. The first part (January 1988-August 2004) is used to estimate the model parameters; the second part (August 2004-May 2009) is used to verify *ex-post* the impact of the forecasted allocation choices.

3.1 *Empirical evidence from the MSCI countries indices*

A preliminary analysis of the MSCI World Index and the other 24 indices suggests that all the log returns are non-Gaussian distributed. This can be seen from Table 1 where the stable maximum likelihood parameters, the mean, the variance, the skewness $\frac{E\left((z-E(z))^3\right)}{E((z-E(z))^2)^{3/2}}$ and kurtosis$\frac{E\left((z-E(z))^4\right)}{E((z-E(z))^2)^2}$ of log returns are reported. In particular, the results reported in the table suggest (1) the returns exibit heavy tails since the stability parameter alpha is always less than 2 and the kurtosis is much higher than 3 and (2) the returns are asymmetric since the skewness parameter and the stable parameter beta are always different from zero.

Therefore it is not surprising that when we consider tests for normality such as the Jarque-Bera and Kolmogorv-Smirnov tests (with a 95% confidence level), the null hypothesis of normality for the daily log returns is rejected for all the stock indices. However if we test the stable Paretian assumption with a 95% confidence level employing the Kolmogorv-Smirnov statistic, the null hypothesis is rejected for only three of the 25 stock indices. If we assume that returns are Student t distributed, the Kolmogorv-Smirnov test rejects the null hypothesis for 11 of the 25 stock indices. Moreover, observing the covariation of the last three years of our study period, we find that the dependence model cannot be approximated with a multivariate normal distribution because it fails to describe the tail dependence and asymmetry of returns.

3.2 *Comparison between three-fund separation models without portfolio constraints*

In our first comparison, we assume the vectors of returns $z_{t_j} = [z_{1,t_j}, ..., z_{24,t_j}]'$ are statistically independent and follow the model given by (1). Moreover, we assume that unlimited short sales are allowed, and we consider a temporal horizon $T = 5$ where the risk-free returns are those corresponding to Libor daily returns at the respective dates of portfolio recalibrations. Thus, every five days we get the optimal strategies for the subsequent five days given by the solution to problem (3). In the empirical comparison, we approximate optimal solutions obtained by maximizing different expected utility functions. In particular, we use those multi-portfolio policy solutions to problem (3) that maximize the sample expected utility. So, we assume that each investor maximizes one from among the following five expected utility functions:

1) $\max\limits_{\{x_{t_j}\}_{j=0,1,...,T-1}} \quad E\log(W_T);$

2) $\max\limits_{\{x_{t_j}\}_{j=0,1,...,T-1}} \quad -E\left(\exp(-\gamma W_T)\right)$ with $\gamma = 1, 5, 7, 17;$

3) $\max\limits_{\{x_{t_j}\}_{j=0,1,...,T-1}} \quad E\left(\frac{W_T^c}{c}\right)$ with $c = -1.5, -2.5;$

4) $\max\limits_{\{x_{t_j}\}_{j=0,1,...,T-1}} \quad E(W_T) - cE\left(|W_T - E(W_T)|^{1.3}\right)$ with $c = 1,\ 2,\ 5;$

5) $\max_{\{x_{t_j}\}_{j=0,1,...,T-1}} E(W_T) - cE\left(|W_T - E(W_T)|^2\right)$ with $c = 1, 2, 5$.

Observe that when the returns are in the domain of attraction of a stable law, with $1 < \alpha_1, \alpha_2 < 2$, the above expected utility functions could be infinite. However, assuming that the returns are truncated far enough, those formulas are formally justified by pre-limit theorems (see Klebanov et al., 2000; Klebanov et al., 2001), which provide the theoretical basis for modeling heavy-tailed bounded random variables with stable distributions. This truncation serves only to justify that expected utility is theoretically finite. It has no real implications since we could consider the truncation far enough to take into account all the historical observations of returns. On the one hand, an investor can always approximate his expected utility since he works with a finite amount of data. On the other hand, pre-limiting theorems suggest that it is the central part ("body") of the distribution that plays a fundamental role in any approximation since finitely many empirical observations can never justify any tail behavior. Thus, if we can better approximate the sum of i.i.d. random variables with a given stable law, the good approximation should be maintained even if the random variable admits all finite moments and even increasing substantially (but not to infinite) the number of observations. In our specific case, even if all returns admit finite all the moments and we observe a better approximation of returns with a stable non-Gaussian law, then W_T should be still well approximated by the stable law, but the expected utility should be finite (i.e., $|E(u(W_T))| < \infty$). Therefore, we are justified in approximating the expected utility $E(u(W_T))$ with the sample expected utility $\frac{1}{N}\sum_{i=1}^{N} u(W_T^{(i)})$ even for a large N.

For our model we need to estimate several parameters: the index of stability α_1, the mean μ_t, the dispersion matrix Q_t, and the vector $b_t = [b_{1,t}, ..., b_{24,t}]'$. In order to simplify our empirical comparison, we assume these parameters are constant over the time $t = 1, ..., 5$. Then every five days we recalibrate them and we use again compute the multi-portfolio policy solutions for problem (3) based on the new estimation of the mean, the dispersion matrix, and the vector b_t. We estimate α_1 to be equal to the mean of 10,000 indexes of stability computed with the maximum likelihood estimator (MLE) of random portfolios of the disturbances $\widetilde{\varepsilon} = z - \mu - \widehat{b}Y$, i.e., $\alpha_1 = \frac{1}{10000}\sum_{k=1}^{10000} \alpha_{(k)} = 1.592$; where $\alpha_{(k)}$ is the index of stability of a random portfolio $(x^{(k)})'\widetilde{\varepsilon}$. The estimator of μ is given by the vector $\widehat{\mu}$ of the sample average. Then, we consider as factor Y the centralized MSCI World Index return. Regressing the centered returns $\widetilde{z}_i = z_i - \widehat{\mu}_i$ $(i = 1, ..., 24)$ on Y, we write the following estimators[4] for $b = [b_1, ..., b_{24}]'$ and Q:

$$\widehat{b}_i = \frac{\sum_{k=1}^{N} Y^{(k)} \widetilde{z}_i^{(k)}}{\sum_{k=1}^{N} \left(Y^{(k)}\right)^2}; \quad i = 1,...,24, \tag{11}$$

$$\text{and} \quad \widehat{Q} = [\widehat{q}_{i,j}]$$

where

$$\widehat{q}_{j,j} = \left(A(p) \frac{1}{N} \sum_{k=1}^{N} \left| \widetilde{\varepsilon}_j^{(k)} \right|^p \right)^{\frac{2}{p}},$$

$$\widehat{q}_{i,j} = \frac{1}{2} \left(\left(A(p) \frac{1}{N} \sum_{k=1}^{N} \left| \widetilde{\varepsilon}_i^{(k)} + \widetilde{\varepsilon}_j^{(k)} \right|^p \right)^{\frac{2}{p}} - \widehat{q}_{j,j} - \widehat{q}_{i,i} \right)$$

$p \in (0, \alpha_1)$, $A(p) = \frac{\Gamma\left(1-\frac{p}{2}\right)\sqrt{\pi}}{2^p \Gamma\left(1-\frac{p}{\alpha}\right)\Gamma\left(\frac{p+1}{2}\right)}$, and $\widetilde{\varepsilon}^{(k)} = \widetilde{z}^{(k)} - \widehat{b}Y^{(k)}$ is the sample disturbance vectors. The entries of the dispersion matrix derive from the moment method suggested by Property 1.2.17 in Samorodnitsky and Taqqu (1994) (see also Ortobelli et al., 2004). In addition, arguing along the same lines as Rachev (1991), Götzenberger et al. (2001), and Tokat et al. (2003), we can explain and prove the asymptotic properties of this estimator. This estimator has been proposed as an alternative to the MLE for stable sub-Gaussian distributions that is not efficient to value the joint covariation of the stable dispersion matrix. Although theoretically we can use any $p \in (0, \alpha_1)$ in the estimation of the dispersion matrix parameters, there exists an optimal value of p that optimizes the approximation. This approximation de-

[4] See Kim et al. (2005) for a discussion of the best estimators of vector b when a heavy-tailed series is assumed.

pends on the historical series of disturbance observations $\left\{\widetilde{\varepsilon}_j^{(k)}\right\}_{k=1}^{N}$ (see Lamantia et al., 2006; Rachev, 1991). According to the analysis proposed by Lamantia et al. (2006), we consider the optimal $\widehat{p_j}$ that (during a period of S=329 days, subsequent the *K-th* observation, and considering a window of K=4,000 historical observations) minimizes the average of distance between $\widehat{q}_{j,j,t}(p) = \left(A(p)\frac{1}{N}\sum_{k=1}^{K}\left|\widetilde{\varepsilon}_j^{(t-k)}\right|^p\right)^{1/p}$ and the MLE $\overline{v}_{j,j,t}$ of dispersion (t=1,...,S). That is,

$$\widehat{p_j} = \arg\left(\min_p \frac{1}{S}\sum_{t=1}^{S}\left|\widehat{q}_{jj,t}(p) - \overline{v}_{j,j,t}\right|\right), \quad j = 1, ..., 24.$$

Then we use the common parameter $\widehat{p}$ equal to the mean of the optimal $\widehat{p_j}$, i.e., $\widehat{p} = \frac{1}{24}\sum_{j=1}^{24}\widehat{p_j} \simeq$ 0.49714.

In our comparisons, we use a window of $N = 4,329$ historical or simulated observations. In the simulation approaches, it is necessary to simulate the final wealth after five days:

$$W_5 = W_0\prod_{k=0}^{4}(1 + z_{0,t_k}) + \sum_{i=0}^{3} x'_{t_i}p_{t_i}\prod_{k=i+1}^{4}(1 + z_{0,t_k}) + x'_{t_4}p_{t_4}.$$

Since the vectors of excess returns p_{t_i} are independent, we simulate every T=*5* days, five times N i.i.d. asymmetric random variables Y and five times N i.i.d. vectors of disturbance ε. Then we use formula (1) to generate the returns z_{t_j} and so we compute the vectors of excess returns p_{t_i} i=0,1,2,3,4. In particular, we assume that the random variable Y admits distribution $S_{\alpha_2}(\sigma_Y, \beta_Y, 0)$ where the stable parameters α_2, σ_Y, β_Y are those estimated every T=*5* days on the MSCI World Index. We use the method of Chambers et al. (1976) to generate stable random variables. Then we assume that the vector of disturbance is distributed either with a α_1 stable multivariate sub-Gaussian distribution or with a multivariate Student t distribution with five degree of freedom (this number of degrees of freedom is the approximated integer number of the mean of MLE marginal estimates of degrees of freedom). Observe that both these elliptical vectors (stable sub-Gaussian and multivariate Student t) admit the form:

$$\varepsilon = \sqrt{X}\mathbf{Z}, \tag{12}$$

where the vector Z that is normally distributed with zero mean and covariance matrix Σ and the random variable X is a positive random variable (called a *subordinator*) independent of the vector Z (see, among others, Rachev and Mittnik, 2000). When ε is distributed as a Student t with v degrees of freedom, the subordinator X is distributed as an inverse-chi-square distribution with v degrees of freedom (i.e., X is a random variable $\frac{v}{\chi_v^2}$ where χ_v^2 is a chi-square distribution with v degrees of freedom). In this case the matrix Σ is given by $\Sigma = Q_t \dfrac{v-2}{v}$ where Q_t is the variance covariance matrix of the vector of disturbances ε. Instead, when the vector of disturbances ε is distributed as an α_1 stable multivariate sub-Gaussian with dispersion matrix Q_t and null mean, then the subordinator $X \stackrel{d}{=} S_{\alpha_1/2}\left(2\left(\cos\left(\frac{\pi\alpha_1}{4}\right)\right)^{2/\alpha_1}, 1, 0\right)$ is $\alpha_1/2$ stable distributed with scale parameter $2\left(\cos\left(\frac{\pi\alpha_1}{4}\right)\right)^{2/\alpha_1}$ and skewness parameter $\beta = 1$. In this case the matrix Σ is equal to the dispersion matrix Q_t of α_1 stable sub-Gaussian vector.

The generation of these elliptical distributed vectors is simply obtained by generating scenarios of the vector normally distributed Z and generating scenarios of subordinators. Then the scenarios of disturbance is obtained by (12) (see Rachev and Mittnik, 2000; Cont and Tonkov, 2004, for the generation of subordinated processes).

In order to compare the different models, we use (in a multi-period context) the same algorithm proposed by Giacometti and Ortobelli (2004) and Ortobelli et al. (2005). Thus, first we consider the optimal strategies for different levels of the mean and skewness. Second, we select the portfolio strategies on the efficient frontiers that maximize some parametric expected utility functions for different risk-aversion coefficients. Finally, we compare the performance of the stable Paretian and of moment-based approaches for each optimal allocation proposed.

Therefore, every five trading days during the period 8/10/2004 to 5/26/2009 (1,250 trading days) and considering a window of $N = 4,329$ i.i.d. (historical or simulated) observations $z^{(i)}$ ($i = 1, \ldots, N$) of the vector $z_t = [z_{1,t}, z_{2,t}, ..., z_{24,t}]'$, the main steps in our comparison are the following:

***Step* 1** Consider the optimal portfolio strategies

$$\begin{gathered}
x_j(\lambda_1, \lambda_2) = \left(\tfrac{2}{\alpha_1}\right)^{\frac{1}{(\alpha_1-1)}} \frac{\left(\left(\lambda_1 E(p_{t_j})+\lambda_2 b_{t_j}\right)' Q^{-1}\left(\lambda_1 E(p_{t_j})+\lambda_2 b_{t_j}\right)\right)^{\frac{2-\alpha_1}{(\alpha_1-1)2}}}{B_{j+1}} \times \\
\times Q^{-1}\left(\lambda_1 E(p_{t_j}) + \lambda_2 b_{t_j}\right) \qquad \forall j = 0, 1, 2, 3 \\
x_4(\lambda_1, \lambda_2) = \left(\left(\lambda_1 E(p_{t_4}) + \lambda_2 b_{t_4}\right)' Q^{-1}\left(\lambda_1 E(p_{t_4}) + \lambda_2 b_{t_4}\right)\right)^{\frac{2-\alpha_1}{(\alpha_1-1)2}} \times \\
\times \left(\tfrac{2}{\alpha_1}\right)^{\frac{1}{(\alpha_1-1)}} Q^{-1}\left(\lambda_1 E(p_{t_4}) + \lambda_2 b_{t_4}\right),
\end{gathered}$$

that generate the efficient frontier.

***Step* 2** Choose a utility function u with a given coefficient of aversion to risk.

***Step* 3** Compute for every multi-period efficient frontier

$$\max_{\lambda_1,\lambda_2} \tfrac{1}{N} \textstyle\sum_{i=1}^{N} u\left(W_5^{(i)}\right),$$

where $W_5^{(i)} = \prod_{k=0}^{4}(1 + z_{0,k}) + \sum_{j=0}^{3} x_j'(\lambda_1, \lambda_2) p_j^{(i)} \prod_{k=j+1}^{4}(1 + z_{0,k}) + x_5'(\lambda_1, \lambda_2) p_4^{(i)}$ is the *i-th* observation (or scenario) of the final wealth and $p_t^{(i)} = [p_{1,t}^{(i)}, ..., p_{n,t}^{(i)}]'$ is the *i-th* historical or simulated observation of the vector of excess returns $p_{k,t}^{(i)} = z_{k,t}^{(i)} - z_{0,t}$ relative to the *t-th* period. In particular, we implicitly assume the approximation:

$$\frac{1}{N}\sum_{i=1}^{N} u\left(W_5^{(i)}\right) \approx E\left(u\left(W_5^{(i)}\right)\right)$$

and that $\{x_j(\lambda_1, \lambda_2)\}_{j=0,1,\ldots,4}$ are the optimal portfolio strategies of formula (4).

***Step* 4** Repeat Steps 2 and 3 for every utility function and for every risk-aversion coefficient for the entire period August 2004-May 2009.

Tables 2 and 3 show the results obtained respectively with historical and simulated data. In both tables we report (1) an average of the maximum expected utility obtained during the period August 2004-May 2009, (2) the *ex-post* final wealth, and (3) a measure of the distance between the optimal portfolio compositions of the two different distributional assumptions. To emphasize the differences in the optimal portfolio composition, we employ the following notation:

a) x_t^{stable} denotes the optimal portfolio policies that realize the maximum expected utility at time t assuming the stable Paretian model;

b) x_t^{moment} denotes the optimal portfolio policies that realize the maximum expected utility at time t assuming the moment-based approach. For the simulated approach x_t^{moment} identifies the optimal portfolio policies obtained by simulating the vectors of disturbances with a multivariate Student t distribution with $v = 5$ degrees of freedom.

Then we consider the average of the absolute difference between the portfolio compositions at each time t, i.e.,

$$\frac{1}{1250}\sum_{t=1}^{1250}\sum_{k=0}^{24}\left|x_{k,t}^{stable} - x_{k,t}^{moment}\right|, \tag{13}$$

where $x_{k,t}^{(.)}$ is the k-th component of optimal porfolio strategy $x_t^{(.)}$. This measure points out how much the portfolio composition for each recalibration changes. Observe that the distance $\rho_t = \sum_{k=0}^{24}\left|x_{k,t}^{stable} - x_{k,t}^{moment}\right|$ is equal to zero only if the portfolio composition is the same for the two approaches. If no short sales are allowed and portfolio components belong to the 25-dimensional simplex (i.e., $\sum_{k=0}^{24} x_{k,t}^{(.)} = 1$), then distance ρ_t should be less than or equal to 2. If no short sales are allowed, $\sum_{k=0}^{24} x_{k,t}^{(.)} = 1$ and $\rho_t = 2$, then the assets used in the two approaches at time t are different (i.e., the portfolio components that are positive for one approach are null for the other one and vice versa). Since in this first comparison unlimited short sales are allowed, then the distance $\sum_{k=0}^{24} \left| x_{k,t}^{stable} - x_{k,t}^{moment} \right|$ could also be greater than 2 (see Table 2) and there could be some assets purchased with one approach and sold with the other one. In the comparison provided in Tables 2 and 3 we distinguish this distance when historical data and simulated data are utilized.

Tables 2 and 3 summarize the comparison between the three-fund separation approaches discussed above. In particular, these tables show (1) that the average of the *ex-ante* optimal solutions that maximize the expected utility functions are always on the mean-dispersion-skewness frontier of the stable Paretian model and (2) investors increase their performance when they use the stable Paretian model. Only in a few cases do we observe that the *ex-post* final wealth of the moment-based model exceeds that of the stable Paretian model. Moreover, we observe substantial differences in the optimal portfolio composition. Considering that the moment-based and stable Paretian models are based on a different risk perception of the disturbances, this empirical comparison suggests that the disturbances have a strong impact on the portfolio selection decisions made by investors. There is not a substantial difference between the model based on historical data and simulated data. However we observe that there is less of an impact on the portfolio composition when we use simulated data.

3.3 *Comparison between three-fund separation models with portfolio constraints*

Now we will compare dynamic strategies with constant and proportional transaction costs of 0.2%[5] when short sales are not permitted. In particular, we assume that the returns follow the conditional model given by (7) and that each investor recalibrates his portfolio daily starting from 8/10/2004 to 5/26/2009. Under these assumptions, we must also estimate the decay factor parameter λ and the conditional dispersion matrix $Q_{t+1/t}$ of vector of disturbances ε, while all the other parameters are estimated as in Section 3.2. When the dispersion matrix

[5] The transaction costs generally change for different countries. Here we fix some indicative transaction costs often used by institutional investors in Italy.

is the variance-covariance matrix then,

$$\begin{aligned}\sigma^2_{ij,t+1/t} &= E_t\left(\left(\widetilde{z}_{i,t+1} - b_{i,t}Y_{t+1}\right)\left(\widetilde{z}_{j,t+1} - b_{j,t}Y_{t+1}\right)\right) \simeq \\ &\simeq (1-\lambda)\sum_{k=0}^{K}\left(\lambda^{K-k}\left(\widetilde{z}_{i,t-K+k} - b_{i,t}Y_{t-K+k}\right)\left(\widetilde{z}_{j,t-K+k} - b_{j,t}Y_{t-K+k}\right)\right),\end{aligned} \tag{14}$$

where $\widetilde{z}_{i,t} = z_{i,t} - \mu_{i,t}$. Instead, the forecasted time $t+1$ stable scale parameter of the *i-th* residual is given by:

$$\begin{aligned}\sigma_{ii,t+1/t} &= \left(E_t(|\widetilde{z}_{i,t+1} - b_{i,t}Y_{t+1}|^p)A(p)\right)^{1/p} \simeq \\ &\simeq \left(A(p)(1-\lambda)\sum_{k=0}^{K}\lambda^{K-k}\left|\widetilde{z}_{i,t-K+k} - b_{i,t}Y_{t-K+k}\right|^p\right)^{1/p}.\end{aligned} \tag{15}$$

The time $t+1$ stable covariation parameter between the *i-th* and the *j-th* residual is defined by $\sigma^2_{ij,t+1/t} = \frac{\left(B_{ij,t+1/t}(p)\right)^{2/p} - \sigma^2_{ii,t+1/t} - \sigma^2_{jj,t+1/t}}{2}$ and

$$\begin{aligned}B_{ij,t+1/t}(p) &= A(p)E_t\left(\left|\left(\widetilde{z}_{i,t+1} - b_{i,t}Y_{t+1}\right) + \left(\widetilde{z}_{j,t+1} - b_{j,t}Y_{t+1}\right)\right|^p\right) \simeq A(p)\times \\ &\times(1-\lambda)\sum_{k=0}^{K}\left(\lambda^{K-k}\left|\left(\widetilde{z}_{i,t-K+k} - b_{i,t}Y_{t-K+k}\right) + \left(\widetilde{z}_{j,t-K+k} - b_{j,t}Y_{t-K+k}\right)\right|^p\right).\end{aligned} \tag{16}$$

In order to estimate the the decay factor parameter λ we use the same methodology proposed in the RiskMetrics model when the disturbance vector admits finite variance (see Longerstaey and Zangari, 1996). However, in the stable Paretian case, we utilize the estimation procedure proposed by Lamantia et al. (2006). Accordingly, we first estimate the decay factors λ_i of each marginal return residuals by minimizing the root mean squared error (RMSE) on the historical series of data. We evaluate for any series the optimal λ_i, that minimizes the RMSE between $\sigma^p_{ii,t/t-1} = A(p)(1-\lambda)\sum_{k=0}^{K}\lambda^{K-k}\left|\widetilde{z}_{i,t-1-K+k} - b_{i,t}Y_{t-1-K+k}\right|^p$ and the power of error $A(p)\left|\widetilde{z}_{i,t} - b_{i,t}Y_t\right|^p$ valued at time t (t=1,...,S), during a period of S=329 days (subsequent the

K-th observation) and considering a window of K=4,000 historical observations. That is,

$$\lambda_i = \arg\left(\min_{\overline{\lambda}}\sqrt{\frac{1}{S}\sum_{t=1}^{S}\left|A(p)\left|\widetilde{z}_{i,t}-b_{i,t}Y_t\right|^p-\sigma^p_{ii,t/t-1}(\overline{\lambda}_i)\right|^2}\right). \tag{17}$$

Thus, we solve optimization problem (17) by discretizing $\overline{\lambda}_i$ with the same steps $\Delta\overline{\lambda}\simeq 0.005$ for every i. The optimal parameter λ,is defined as:

$$\widehat{\lambda}=\sum_{i=1}^{24}\phi_i\lambda_i$$

where

$$\phi_i=\frac{\theta_i}{\sum_{k=1}^{24}\theta_k};\ \theta_i=\frac{\sum_{k=1}^{24}\min_{\overline{\lambda}}\sqrt{\sum_{t=1}^{S}\left|A(p)\left|\widetilde{z}_{k,t}-b_{i,t}Y_t\right|^p-\sigma^p_{kk,t/t-1}(\overline{\lambda})\right|^2}}{\min_{\overline{\lambda}}\sqrt{\sum_{t=1}^{S}\left|A(p)\left|\widetilde{z}_{i,t}-b_{i,t}Y_t\right|^p-\sigma^p_{ii,t/t-1}(\overline{\lambda})\right|^2}}.$$

Once we have the decay factors $\widehat{\lambda}$ for the dispersion of disturbance ε, we can easily compute the conditional dispersion matrix $Q_{t+1/t}$. We apply this procedure daily during all the periods of the *ex-post* comparison.

In order to describe the different portfolio strategies considering transaction costs and short-sale constraints with historical data, we have to determine the optimal choices of investors at each time t. Thus, at each time t, we have to solve two different optimization problems: the first to fit the efficient frontier with transaction cost constraints and the second to determine the optimal expected utility on the efficient frontier. In particular, considering N observations $z^{(i)}$ $(i=1,\ldots,N)$ of the vector $z_t=[z_{1,t},z_{2,t},...,z_{25,t}]'$, the main steps of our comparison are summarized in the following algorithm:

***Step* 1** We choose a utility function u with a given coefficient of aversion to risk.

***Step* 2** At time t_0=8/10/2004, we fit the three-parameter efficient frontiers corresponding to the different distributional hypothesis: moment-based and stable Paretian approaches. Therefore, we fit 5,000 optimal portfolio weights x_{t_0} varying the weekly mean m_W and the index of skewness b^* in the following quadratic programming problem:

$$\begin{array}{c} \min_{x_{t_0}} x'_{t_0} Q x_{t_0} \quad \text{subject} \quad \text{to} \\ x'_{t_0}\mu + (1 - x'_{t_0}e)z_{0,0} = m_{W_{t_0}} \\ x'_{t_0}b_t = b^*, \quad 0 \leq x'_{t_0}e \leq 1 \\ \text{and } x_{i,t_0} \geq 0, \quad i = 1, ..., n \end{array}, \tag{18}$$

and $W_{t_0} = x'z_t + (1 - x'_{t_0}e)z_{0,0}$. We assume that over time t the vector mean $\mu = E(z_t)$ and the dispersion matrix Q of the disturbances are constant. Then, for each efficient frontier, we have to determine the portfolio weights x_{t_0} that maximize the expected utility given by the solution to the following optimization problem

$$\begin{array}{c} \max_{x_{t_0}} \frac{1}{N} \sum_{i=t_0-N}^{t_0} u\left(x'_{t_0} z^{(i)} + (1 - x'_{t_0}e)z_{0,0}\right) \\ \text{subject} \quad \text{to} \\ x_{t_0} \text{ are optimal portfolio of the efficient frontier.} \end{array}$$

Thus given

$$x^*_{t_0} = \arg(\max_{x_{t_0} \text{belongs to the efficient frontier}} (E(u(x'_{t_0}z_t + (1 - x'_{t_0}e)z_{0,0}))))$$

the *ex-post* final wealth at time 5/31/2004 is obtained by $W_1 = W_0(1 + \left(x^*_{t_0}\right)' z^{(t_1)} + (1 - e'x^*_{t_0})z_{0,1} - 0.002)$ where 0.002 is the fixed proportional transaction costs for unity of wealth invested. In order to determine the optimal portfolio strategies for the other periods, we have to take into account that the investor pays proportional transaction costs of 0.2% on the absolute difference of the changes of portfolio compositions. Thus, at time t_k (after k weeks) we fit 5,000 optimal portfolio weights x_{t_k} varying the weekly mean $m \geq z_{0,t_k}$ and the index of skewness b^* in the following optimization problem:

$$\begin{array}{c} \min_{x_{t_k}} x'_{t_k} Q x_{t_k} \quad \text{subject} \quad \text{to} \\ m = E(X(x_{t_k})) \\ x'_{t_k} b_t = b^*, \quad 0 \leq x'_{t_k} e \leq 1 \\ \text{and } x_{i,t_k} \geq 0, \quad i = 1, ..., 25 \end{array},$$

where $X(x_{t_k}) = x'_{t_k} z_{t_k} + (1 - x'_{t_k} e) z_{0,t_k} - t.c.(x_{t_k})$ and $t.c.(x_{t_k})$ represents the transaction costs at time t_k of portfolio x_{t_k} which are given by

$$0.002 \left| (1 - x'_{t_k} e) - \frac{(1 - x'_{t_{k-1}} e)(1 + z_{0,t_k})}{(1 - x'_{t_{k-1}} e)(1 + z_{0,t_k}) + \sum_{i=1}^{24} x_{i,t_{k-1}} (1 + z_i^{(t_k)})} \right| +$$
$$+ 0.002 \sum_{i=1}^{24} \left| x_{i,t_k} - \frac{x_{i,t_{k-1}} (1 + z_i^{(t_k)})}{(1 - x'_{t_{k-1}} e)(1 + z_{0,t_k}) + \sum_{i=1}^{24} x_{i,t_{k-1}} (1 + z_i^{(t_k)})} \right|,$$

where $x_{i,t_{k-1}}(1 + z_i^{(t_k)})$ is the percentage of wealth invested on the *i-th* stock at time t_{k-1}capitalized at time t_k. Therefore, for each efficient frontier (the moment-based and stable Paretian ones), we have to determine the optimal portfolio weights

$$x^*_{t_k} = \arg(\max_{x_{t_k} \text{belongs to the efficient frontier}} (E(u(X(x_{t_k}))))).$$

***Step* 3** We compute the *ex-post* final wealth that is given by

$$W_{t_{k+1}} = W_{t_k} (1 + \left(x^*_{t_k}\right)' z^{(t_k+1)} + (1 - e' x^*_{t_k}) z_{0,t_{k+1}} - t.c. \left(x^*_{t_k}\right)) \tag{19}$$

where the transaction costs $t.c. \left(x^*_{t_k}\right)$ are defined above.

***Step* 4** We repeat Steps 2 and 3 for every utility function and for every risk-aversion coefficient.

Moreover, as in Section 3.2, we can apply the same approach even to simulated data. In the simulated approaches, every day we generate N=4,329 predicted vectors of returns $z^{(i)}$ at time t+1. Thus, we first generate N i.i.d. scenarios of random variable Y with an α_2-stable distribution $S_{\alpha_2}(\sigma_Y, \beta_Y, 0)$ and then we generate N i.i.d. scenarios of the disturbance vector ε_{t+1} either with a multivariate Student t distribution with five degrees of freedom and variance-covariance matrix $Q_{t+1/t}$ (valued with (9)) or with a multivariate α_1 stable sub-Gaussian vector with dispersion matrix $Q_{t+1/t}$ (valued with (10)). Finally, we have N i.i.d. scenarios of future returns $z^{(i)}$ and we compute the maximum sample expected utility solving the following portfolio selection problem:

$$\frac{1}{N}\sum_{i=t+1-N}^{t+1} u\left(x'_{t+1}z^{(i)} + (1 - x'_{t+1}e)z_{0,0} - t.c.(x_{t+1})\right)$$
$$0 \leq x'_{t+1}e \leq 1; \text{and } x_{i,t+1} \geq 0, \quad i = 1, ..., n,$$

where $t.c.(x_t)$ represents the transaction costs at time t of portfolio x_t which are given by the previous formula. The the *ex-post* final wealth is still given by (19).

Observe that at each time t_k the investor's optimal choices are uniquely characterized by the mean, the dispersion, and the skewness. In particular, if we assume that $\alpha = \alpha_1 = \alpha_2$, the vector of returns is jointly α-stable distributed and every centered portfolio $\widetilde{z}_{p,t_k} = \sum_{i=1}^{n} x_{i,t_k}\widetilde{z}_{i,t_k}$ admits the stable distribution $S_\alpha(\sigma_{p,t_k}, \beta_{p,t_k}, 0)$ where $\sigma_{p,t_k} = \left(\left(x'_{t_k}Qx_{t_k}\right)^{\alpha/2} + \left|x'_{t_k}b\sigma_Y\right|^\alpha\right)^{1/\alpha}$ is the volatility and $\beta_{p,t_k} = \frac{\left|x'_{t_k}b\sigma_Y\right|^\alpha sgn(x'_{t_k}b)\beta_Y}{\left(x'_{t_k}Qx_{t_k}\right)^{\alpha/2} + \left|x'_{t_k}b\sigma_Y\right|^\alpha}$ is the portfolio skewness. Thus, we can represent the investor's optimal choices in terms of the mean $E(x'_{t_k}z + (1 - x'_{t_k}e)z_{0,t_k} - t.c.(x_{t_k}))$, the dispersion σ_{p,t_k}, and the portfolio skewness β_{p,t_k}. Similarly, when we consider the moment-based model, the optimal portfolio choices can be represented in terms of the mean, the standard deviation, and the Fisher skewness parameter given by $\frac{E\left((\widetilde{z}_{p,t_k} - E(\widetilde{z}_{p,t_k}))^3\right)}{E\left((\widetilde{z}_{p,t_k} - E(\widetilde{z}_{p,t_k}))^2\right)^{3/2}}$ (if we implicitly assume that the returns admit finite the third moment, i.e. the disturbance vector and the random varaible Y have finite the first three moments).

Figure 1 shows the three-dimensional efficient frontiers for the two models. As expected, in both cases the optimal choices are represented by a curved plane. First of all, we observe that the average of *ex-ante* expected utility obtained with the stable Paretian approach is always greater than that obtained with the moment-based model, a result that holds for almost all experiments except in few cases when we use simulated data (see Tables 4 and 5). Even in this comparison, however, we consider the distance given by (13) between the portfolio compositions at each time t_j. Then we observe significant differences in the optimal portfolio compositions, although these differences are lower than those obtained when unlimited short sales are allowed. The simulated approach (see Table 5) confirms the approach based on historical series even if we observe a lower average distance between the moment-based optimal portfolios and the stable Paretian one. In contrast, when we plot the distance $\rho_t = \sum_{k=0}^{24} \left| x_{k,t}^{stable} - x_{k,t}^{moment} \right|$, we observe high variability in the distances of optimal portfolios as shown in Figure 2. From the figure we see that the transaction costs could have an important impact on the final wealth.

However, the *ex-post* comparison shows that the final wealths obtained with the stable Paretian model are almost always greater than those obtained with the moment-based model. Practically, as can be seen from Figure 3, in many of the cases investigated the stable Paretian portfolio strategy dominates the moment-based one. The figure shows the *ex-post* final wealth sample path for an investor with exponential utility function. In this figure we also observe the impact of the global financial crisis on the MSCI developed countries during the period November 2007-March 2009. This performance analysis confirms the importance of properly evaluating the disturbance distribution behavior in the fund-separation portfolio models.

3.4 *Backtesting of VaR models based on conditional distributions*

Here we will assess the reliability of the conditional models proposed to compute VaR. We use the same dataset and models used for the portfolio comparison. Thus we compare the performance in predicting VaR when we assume that the random variable Y admits distribution $S_{\alpha_2}(\sigma_Y, \beta_Y, 0)$ where the stable parameters α_2, σ_Y, β_Y are those estimated for the MSCI World Index. Moreover, we assume that the vector of disturbance is either conditional distributed with a α_1 stable multivariate sub-Gaussian distribution or with a conditional multivariate Student t distribution with five degrees of freedom. Thus we compute the interval forecasts considering two different confidence level $\theta = 95\%$ and $\theta = 99\%$ for 150 randomly chosen portfolios $z_{p,t}$. All parameters are computed as explained in Section 3.2.

Since in this case the return distributions are defined by their characteristic function (8), then at each time t from 8/10/2004 to 5/26/2009 we determine the VaR simulating 100,000 future scenarios of the conditional ditributions and taking the respective percentiles of the simulated data.

We began by determining for any portfolio how many times during the *ex-post* period 8/10/2004 to 5/26/2009 the profits/losses fall outside the confidence interval. In particular, for $\theta = 95\%$ and $\theta = 99\%$, the expected number of observations outside the confidence interval must not exceed 5% and 1%, respectively. The average values obtained among all portfolios for the two distributional assumptions are respectively: (1) 5.9104% when $\theta = 95\%$ and 1.06453% when $\theta = 99\%$ if we assume that the vector of disturbances is conditional stable Paretian distributed and (2) 7.18667% when $\theta = 95\%$ and 1.91627% when $\theta = 99\%$ if we assume that the vector of disturbances is conditional Student t distributed. Thus both models underestimate the number of observations which falls outside the forecast interval, and this effect is more evident when the percentiles are low for the approach based on the Student t distributed residuals. Thus we confirm that the empirical distribution tails are even fatter than the models could predict.

Moreover, we test with 95% confidence level the VaR prediction using the unconditional and conditional coverage tests proposed by Kupiec (1995) and Christoffersen (1998). These tests, summarized in the table below, confirm the previous results.

No. of portfolios for which VaR is acceptably accurate

	Stable Paretian	disturbance	Student t	disturbance
	$\theta = 95\%$	$\theta = 99\%$	$\theta = 95\%$	$\theta = 99\%$
Conditional coverage test	146	150	143	148
Unconditional coverage test	140	148	135	144

All the empirical results confirm that the stable Paretian assumption allows for a better prediction of the potential losses than models based on higher moments.

4. Conclusions

In this paper, we examine a stable Paretian version of the three-fund separation model and propose VaR models with stable distributed returns. We first discuss portfolio choice models considering returns with heavy-tailed distributions. In order to present heavy-tailed models that consider the asymmetry of returns, we examine a discrete-time three-fund separation model where the portfolios are in the domain of attraction of a stable law. Second, we compare the portfolio selection perfomance under different distributional assumptions from the perspective of different non-satiable risk-averse investors, using both historical and simulated data. Our empirical comparison demonstrates that heavy tails of disturbances can have a fundamental impact on the asset allocation decisions by investors. Because the stable Paretian model takes into account the heavy tails of disturbances, we find that the stable Paretian model dominates the moment-based model in terms of expected utility and *ex-post* final wealths. These results hold for both historical and simulated data. Finally, we assess the reliability of the conditional models proposed to compute VaR. The empirical comparison confirms that when the percentiles are below 5%, the stable Paretian model provides a greater ability to predict future losses than models with thiner tails.

5. Appendix

Under the assumption of the model presented in Section 2.1 any risk-averse investor will choose solutions to problem (3) if unlimited short sales are allowed. As a matter of fact, recall that all risk-averse investors (i.e., investors with concave utility functions) prefer the return X to the return Z if and only if X dominates Z in the sense of Rothschild-Stiglitz (see Rothschild and Stiglitz, 1970) or equivalently if and only if $E(X)$=$E(Z)$ and

$$\int_{-\infty}^{v} \Pr\left(X \leq s\right) ds \leq \int_{-\infty}^{v} \Pr\left(Z \leq s\right) ds$$

for every real v. Let W_x and W_y be two admissible final wealths determined respectively by the portfolio policies x_{t_j} and y_{t_j}. Suppose, under the assumptions of model (1) that W_x and W_y have the same mean $E(W_x) = E(W_y)$ and the same parameter $A_x = A_y$. Then we have the following equality in distribution (conditioned at $Y = u$) for any real u

$$X_{/Y=u} = \frac{W_x - E(W_x) - A_x u}{\sigma_{\left(x'_{t_i}\varepsilon_{t_i}\right)}} \stackrel{d}{=} \frac{\Psi_x}{\sigma_{\left(x'_{t_i}\varepsilon_{t_i}\right)}} \stackrel{d}{=} \frac{\Psi_y}{\sigma_{\left(y'_{t_i}\varepsilon_{t_i}\right)}} \stackrel{d}{=} S_{\alpha_1}\left(1,0,0\right).$$

Let's suppose that $\sigma_{\left(x'_{t_i}\varepsilon_{t_i}\right)} > \sigma_{\left(y'_{t_i}\varepsilon_{t_i}\right)}$. Then, W_y dominates W_x in the sense of Rothschild-Stiglitz because for every real v :

$$\begin{aligned}
&\textstyle\int_{-\infty}^{v} \left(\Pr\left(W_y \leq s\right) - \Pr\left(W_x \leq s\right)\right) ds = \\
&= \textstyle\int_{-\infty}^{v} \int_R \left(\Pr\left(X \leq \frac{s-E(W_y)-A_y u}{\sigma_{\left(y'_{t_i}\varepsilon_{t_i}\right)}} \middle| Y = u \right) - \right. \\
&\left. - \Pr\left(X \leq \frac{s-E(W_y)-A_x u}{\sigma_{\left(x'_{t_i}\varepsilon_{t_i}\right)}} \middle| Y = u \right)\right) f_Y(u)du\ ds = \\
&= \textstyle\int_R \int_{-\infty}^{v} \left(\Pr\left(X \leq \frac{s-E(W_y)-A_y u}{\sigma_{\left(y'_{t_i}\varepsilon_{t_i}\right)}} \middle| Y = u \right) - \right. \\
&\left. - \Pr\left(X \leq \frac{s-E(W_y)-A_x u}{\sigma_{\left(x'_{t_i}\varepsilon_{t_i}\right)}} \middle| Y = u \right)\right) ds f_Y(u)du \leq 0
\end{aligned}$$

where f_Y is the density of Y. Therefore, the non-dominated portfolio policies are obtained by minimizing the disturbance dispersion $\sigma_{\left(x'_{t_i}\varepsilon_{t_i}\right)}$ for some fixed mean $E(W_x)$ and parameter A_x.

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	STABLE		PARAMETERS					
	alpha	**beta**	**sigma**	**mu**	**mean**	**st.dev**	**skewness**	**kurtosis**
World	1.6276	-0.1448	0.0047	0.00009	0.00020	0.0091	-0.17182	12.7691
Australia	1.6878	-0.0789	0.0072	0.00033	0.00031	0.0134	-0.61244	13.4278
Austria	1.5878	-0.0761	0.0070	0.00026	0.00026	0.0147	0.00068	15.6804
Belgium	1.6232	-0.1705	0.0066	0.00000	0.00022	0.0129	-0.24863	13.6295
Canada	1.4980	-0.1199	0.0056	0.00021	0.00034	0.0123	-0.62201	16.3169
Denmark	1.6711	-0.1290	0.0070	0.00040	0.00048	0.0129	-0.20411	11.3115
Finland	1.5910	0.0250	0.0107	0.00058	0.00048	0.0206	-0.05847	9.6749
France	1.7007	-0.1805	0.0075	0.00017	0.00036	0.0137	0.09430	11.1642
Germany	1.6565	-0.1631	0.0079	0.00014	0.00035	0.0148	-0.06716	10.2836
Greece	1.5996	0.0668	0.0098	0.00055	0.00043	0.0188	0.24715	9.0996
Honk Kong	1.5218	-0.0131	0.0078	0.00048	0.00043	0.0163	-0.36266	18.8094
Ireland	1.5692	-0.0845	0.0072	0.00009	0.00015	0.0151	-0.48533	15.9224
Italy	1.7011	-0.0862	0.0082	0.00010	0.00020	0.0147	0.07116	9.7224
Japan	1.7216	0.0673	0.0087	0.00007	0.00005	0.0149	0.32450	7.8281
Malaysia	1.3611	0.0096	0.0061	0.00035	0.00033	0.0175	0.75472	58.0064
Netherlands	1.6177	-0.1135	0.0066	0.00019	0.00031	0.0130	-0.00965	11.8888
New Zealand	1.7296	-0.1206	0.0082	-0.00006	0.00005	0.0144	-0.23130	9.8970
Norway	1.6157	-0.0559	0.0083	0.00047	0.00040	0.0166	-0.20376	13.3660
Portugal	1.6226	0.0108	0.0064	0.00012	0.00011	0.0124	0.04648	12.3495
Singapore	1.5332	-0.0504	0.0067	0.00019	0.00033	0.0139	0.11876	10.8568
Spain	1.6759	-0.1241	0.0077	0.00018	0.00034	0.0142	0.08173	10.9757
Sweden	1.6347	-0.0878	0.0090	0.00026	0.00048	0.0170	0.24015	9.3579
Switzerland	1.7191	-0.1393	0.0067	0.00027	0.00039	0.0118	0.02871	9.0023
UK	1.6351	-0.1469	0.0064	0.00001	0.00021	0.0123	0.12680	14.0359
USA	1.4470	-0.0470	0.0052	0.00027	0.00030	0.0113	-0.07017	13.0815

Table 1 MLE stable parameters, mean, standard deviation, skewness $E((z-E(z))^3)/E((z-E(z))^2)^{3/2}$ and kurtosis $E((z-E(z))^4)/E((z-E(z))^2)^2$ assuming daily return series between January 1988 and August 2004.

Expected Utility	Stable Paretian unconditional model With historical series Expected Utility	Final Wealth	Moment-based unconditional model With historical series Expected Utility	Final Wealth	Difference between portfolio compositions $\frac{1}{1250}\sum_{j=1}^{1250}\sum_{i=0}^{24}\left\|x_{i,t_j}^{stable}-x_{i,t_j}^{moment}\right\|$
E(log(X))	**0.00000064**	**1.14435**	-1.2E-09	1.02256	1.11180
-E(exp(-X))	**-0.36858**	**1.14857**	-0.36897	1.03268	1.47680
-E(exp(-5X))	**-0.006842**	**1.21769**	-0.006891	1.01556	2.43240
-E(exp(-7X))	**-0.000912**	**1.10931**	-0.000979	1.00031	2.23340
-E(exp(-17X))	**-0.000000041**	**1.20151**	-0.000000048	1.17264	1.00520
$\frac{-1}{1.5}E\left(X^{-1.5}\right)$	**-0.667592**	**1.06554**	-0.668342	0.98961	1.16630
$\frac{-1}{2.5}E\left(X^{-2.5}\right)$	**-0.401254**	**1.20402**	-0.401853	0.97008	1.22310
$E(X)-E(\|X-E(X)\|^{1.3})$	**1.000747**	**1.24569**	1.000702	1.20359	0.10560
$E(X)-2E(\|X-E(X)\|^{1.3})$	**1.000638**	**1.28422**	1.000611	1.20367	0.09600
$E(X)-5E(\|X-E(X)\|^{1.3})$	**1.000428**	**1.28458**	1.000347	1.20193	0.09240
$E(X)-E(\|X-E(X)\|^{2})$	**0.99953**	1.01079	0.998921	**1.02721**	1.09560
$E(X)-2E(\|X-E(X)\|^{2})$	**0.99896**	1.12821	0.998388	**1.21311**	1.24110
$E(X)-5E(\|X-E(X)\|^{2})$	**1.000351**	1.14820	1.000263	**1.23109**	0.86420

Table 2 Comparison of three parametric efficient frontiers and analysis of each model's performance. We maximize the expected utility on the *ex-ante* efficient frontiers for 24 country equity market indices and 30-day Eurodollar CD. We consider two different models applied to the historical series and assuming returns i.i.d. distributed. We report (1) the average of maximum expected utility obtained during the period August 2004–May 2009; (2) the *ex-post* final wealth of the investor's choices on data 05/26/09; (3) the average of absolute difference of portfolio compositions during the period August 2004–May 2009.

Expected Utility	Stable Paretian unconditional model With simulated scenarios Expected Utility	Final Wealth	Unconditional model With simulated t-Student residuals Expected Utility	Final Wealth	Difference between portfolio compositions $\frac{1}{1250}\sum_{j=1}^{1250}\sum_{i=0}^{24}\left\|x_{i,t_j}^{stable}-x_{i,t_j}^{moment}\right\|$
E(log(X))	**0.00000012**	**1.14865**	-0.0000071	1.04115	0.76436
-E(exp(-X))	**-0.36859414**	**1.13613**	-0.368621741	1.03117	0.98579
-E(exp(-5X))	**-0.00681562**	**1.17866**	-0.006839689	1.00246	2.16559
-E(exp(-7X))	**-0.00093550**	**1.11008**	-0.000937206	1.00108	1.90350
-E(exp(-17X))	-0.000000038	**1.16683**	**-0.000000036**	1.13373	0.96193
$\frac{-1}{1.5}E\left(X^{-1.5}\right)$	**-0.66768720**	**1.03348**	-0.667694974	1.00366	0.72311
$\frac{-1}{2.5}E\left(X^{-2.5}\right)$	**-0.40121167**	**1.18882**	-0.401237846	1.11554	0.55839
$E(X)-E(\|X-E(X)\|^{1.3})$	**1.00075131**	**1.25384**	1.000719652	1.24862	0.04693
$E(X)-2E(\|X-E(X)\|^{1.3})$	**1.00063640**	**1.26792**	1.000609442	1.24382	0.04898
$E(X)-5E(\|X-E(X)\|^{1.3})$	**1.00041810**	**1.23477**	1.000400008	1.21180	0.07499
$E(X)-E(\|X-E(X)\|^{2})$	**0.99953930**	1.09185	0.999532625	**1.12721**	0.48000
$E(X)-2E(\|X-E(X)\|^{2})$	**0.99894770**	1.12342	0.998946553	**1.19535**	0.69721
$E(X)-5E(\|X-E(X)\|^{2})$	**1.00036440**	1.19368	1.000364177	**1.21239**	0.45379

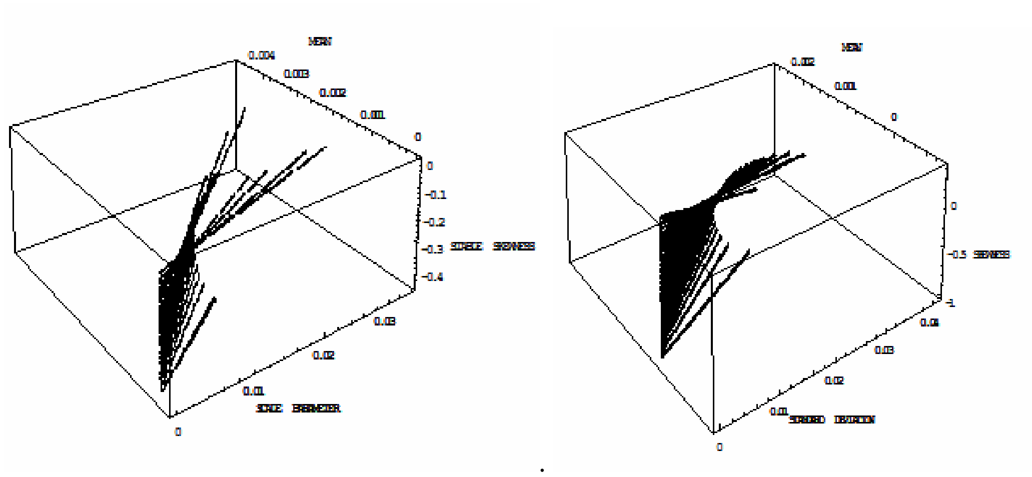
Table 3 Comparison of three parametric efficient frontiers and analysis of each model's performance. We maximize the expected utility on the *ex-ante* efficient frontiers for 24 country equity market indices and 30-day Eurodollar CD. We consider two different models applied to simulated series and assuming returns i.i.d. distributed. We report (1) the average of maximum expected utility obtained during the period August 2004–May 2009; (2) the *ex-post* final wealth of the investor's choices on data 05/26/09; (3) the average of absolute difference of portfolio compositions during the period August 2004–May 2009.

Expected Utility	Stable Paretian conditional model With historical series Expected Utility	Final Wealth	Moment-based conditional model With historical series Expected Utility	Final Wealth	Difference between portfolio compositions $\frac{1}{1250}\sum_{j=1}^{1250}\sum_{i=0}^{24}\left\|x_{i,t_j}^{stable}-x_{i,t_j}^{moment}\right\|$
E(log(X))	**0.000517**	**1.14050**	0.00049	1.03685	0.58820
-E(exp(-X))	**-0.367784**	1.04142	-0.36779	**1.04572**	0.58600
-E(exp(-5X))	**-0.006795**	**1.01822**	-0.00680	0.95283	0.96530
-E(exp(-7X))	**-0.000932**	**1.11465**	-0.00096	1.01324	0.88630
-E(exp(-17X))	**0.000000**	**1.17776**	0.00000	1.13836	0.39890
$\frac{-1}{1.5}E\left(X^{-1.5}\right)$	**-0.666396**	**1.01058**	-0.66650	0.97586	0.92570
$\frac{-1}{2.5}E\left(X^{-2.5}\right)$	**-0.399767**	**0.97932**	-0.39988	0.96567	0.97070
$E(X)-E(\|X-E(X)\|^{1.3})$	**1.000241**	**1.20864**	1.00024	**1.21667**	0.02800
$E(X)-2E(\|X-E(X)\|^{1.3})$	**1.000231**	**1.20834**	1.00023	1.17690	0.02540
$E(X)-5E(\|X-E(X)\|^{1.3})$	**1.000211**	**1.20879**	1.00021	1.15535	0.02450
$E(X)-E(\|X-E(X)\|^{2})$	**1.000475**	1.01273	1.00041	**1.01292**	0.86950
$E(X)-2E(\|X-E(X)\|^{2})$	**1.000386**	0.99321	1.00029	**1.04218**	0.98500
$E(X)-5E(\|X-E(X)\|^{2})$	**1.000289**	1.14942	1.00021	**1.21863**	0.68590

Table 4 Comparison of three parametric efficient frontiers and analysis of each model's performance. We maximize the expected utility on the *ex-ante* efficient frontiers for 24 country equity market indices and 30-day Eurodollar CD. We consider two different models applied to the historical series and assuming residuals are conditional stable or Student *t* distributed. We report (1) the average of maximum expected utility obtained during the period August 2004–May 2009; (2) the *ex-post* final wealth of the investor's choices on data 05/26/09; (3) the average of absolute difference of portfolio compositions during the period August 2004–May 2009.

Expected Utility	Stable Paretian conditional model With simulated scenarios Expected Utility	Final Wealth	Conditional model With simulated t-Student residuals Expected Utility	Final Wealth	Difference between portfolio compositions $\frac{1}{1250}\sum_{j=1}^{1250}\sum_{i=0}^{24}\left\|x_{i,t_j}^{stable}-x_{i,t_j}^{moment}\right\|$
E(log(X))	**0.000511**	1.05661	0.00050	**1.06412**	0.46720
-E(exp(-X))	**-0.367684**	**1.06036**	-0.36789	1.03113	0.43200
-E(exp(-5X))	**-0.006777**	**1.05028**	-0.00682	0.96376	0.84530
-E(exp(-7X))	**-0.000951**	**1.12034**	-0.00099	1.01833	0.67590
-E(exp(-17X))	**0.000000**	**1.14902**	0.00000	1.11555	0.42110
$\frac{-1}{1.5}E\left(X^{-1.5}\right)$	**-0.663513**	**1.04661**	-0.66958	0.96651	0.74810
$\frac{-1}{2.5}E\left(X^{-2.5}\right)$	**-0.399841**	**0.99166**	-0.39990	0.83960	0.88560
$E(X)-E(\|X-E(X)\|^{1.3})$	**1.000265**	**1.20077**	1.00025	1.12465	0.04590
$E(X)-2E(\|X-E(X)\|^{1.3})$	1.000231	**1.22386**	**1.00025**	1.13890	0.04650
$E(X)-5E(\|X-E(X)\|^{1.3})$	1.000208	**1.25755**	**1.00021**	1.14593	0.05180
$E(X)-E(\|X-E(X)\|^{2})$	**1.000479**	**0.93754**	1.00043	0.92306	0.75920
$E(X)-2E(\|X-E(X)\|^{2})$	**1.000341**	0.99747	1.00031	**1.05769**	0.88540
$E(X)-5E(\|X-E(X)\|^{2})$	**1.000320**	1.10562	1.00021	**1.23741**	0.48930

Table 5 Comparison of three parametric efficient frontiers and analysis of each model's performance. We maximize the expected utility on the *ex-ante* efficient frontiers for 24 country equity market indices and 30-day Eurodollar CD. We consider two different models applied to simulated series and assuming residuals are conditional stable or Student *t* distributed. We report (1) the average of maximum expected utility obtained during the period August 2004–May 2009; (2) the *ex-post* final wealth of the investor's choices on data 05/26/09; (3) the average of absolute difference of portfolio compositions during the period August 2004–May 2009.



Mean-scale parameter-stable skewness efficient frontier with the risk-free asset

Mean-standard deviation-skewness efficient frontier with the risk-free asset

Figure 1: Mean-Risk-Skewness efficient frontiers when the risk-free asset is allowed.

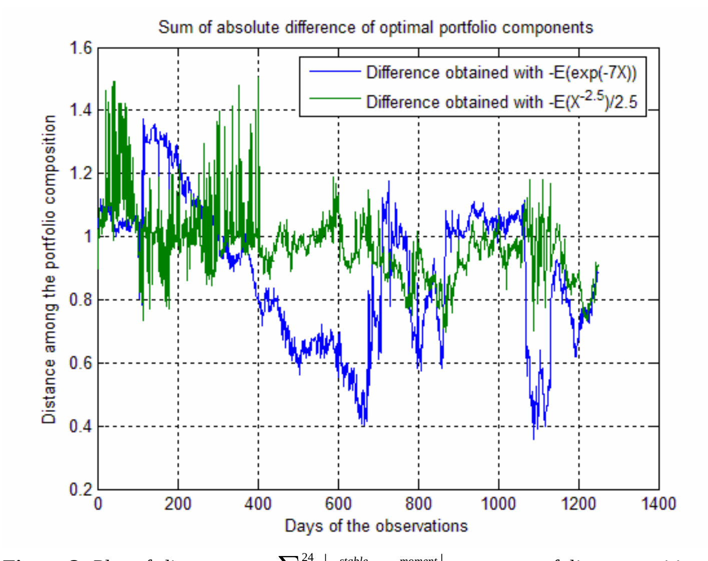


Figure 2: Plot of distance $\rho_t = \sum_{k=0}^{24} \left| x_{k,t}^{stable} - x_{k,t}^{moment} \right|$ among portfolio composition during the period August 2004–May 2009. The optimal portfolios are those obtained maximizing the expected utilities -E(exp(-7X)); -E($X^{-2.5}$)/2.5 on the efficient frontiers of the conditional stable Paretian model and the moment based one.

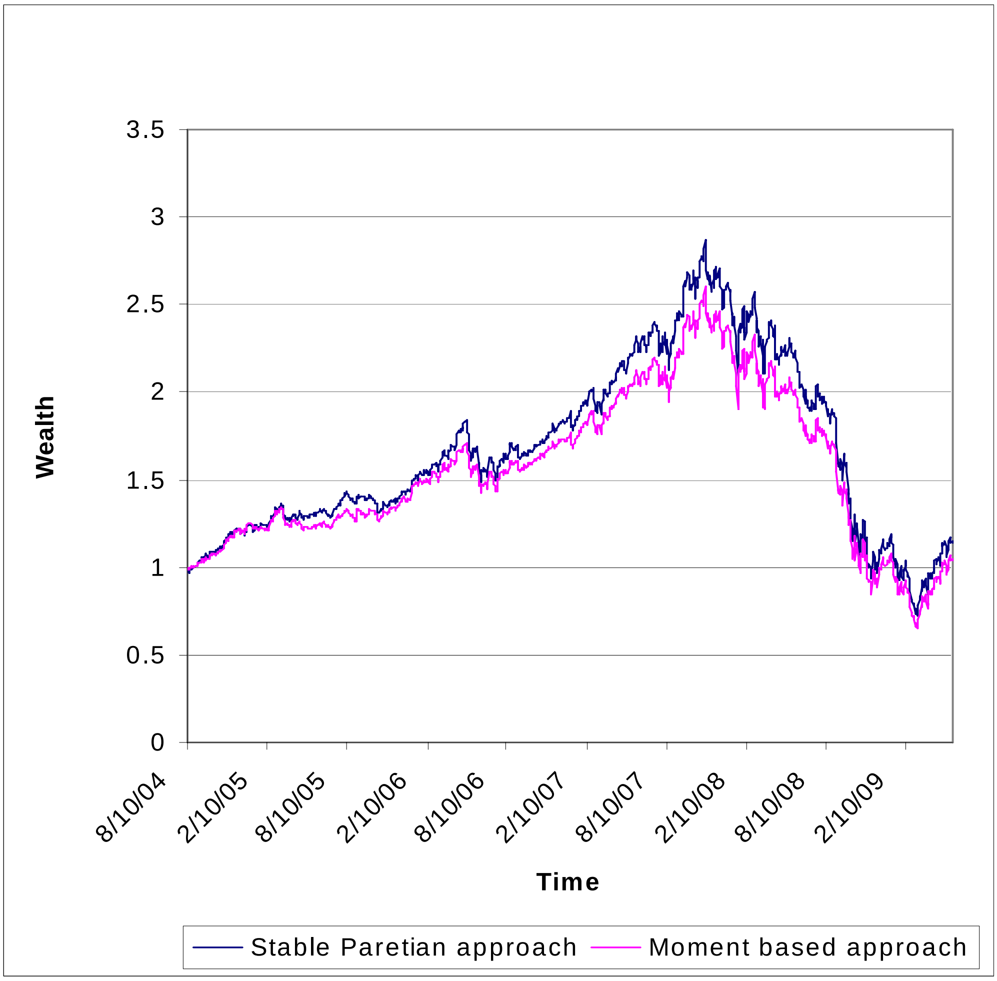


Figure 3: Ex-post comparison of portfolio strategies of an investor with utility function $u(x) = -E(\exp(-X))$ when we consider the conditional stable Paretian and moment based approaches and we use the historical data.